# Asymptotic Behavior of Random Defective Parking Functions

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**Abstract**

Suppose that $m$ drivers each choose a preferred parking space in a linear car park with $n$ spots. In order, each driver goes to their desired spot and parks there if possible. If the spot is already occupied then the car parks in the first available spot after that; if no such spot is available then the car leaves the street without parking. When $m > n$, there will always be defects–cars that are not able to park. Building upon the work in Cameron et al. "Counting defective parking functions,"[1] we introduce a multi-shuffle construction to defective parking functions and investigate parking statistics of a defective parking function chosen uniformly at random.

**Keywords:** Parking function, Multi-shuffle, Asymptotic expansion

## 1 INTRODUCTION

The study of parking functions began when Konheim and Weiss[2] investigated linear probes of hashing storage structures in computer science. Since then, mathematical researchers have applied the concept of parking functions in many intersecting disciplines such as combinatorics, probability, and group theory. The parking problem has counterparts in the enumerative theory of labeled trees and forests[3], in the analysis of non-crossing partitions and hyperplane arrangements,[4;5] in the configuration of abelian sandpiles,[6] among others. We refer to Yan[7] for a comprehensive survey.

The general structure of a parking function problem takes the following form: Consider a parking lot with $n$ parking spots placed sequentially along a one-way street. A line of $m$ cars enters the lot, one by one. The $i$th car drives to its preferred spot $\pi_i$ and parks there if possible. If the spot is already occupied then the car parks in the first available spot after that; if no such spot is available then the car leaves the street without parking.

Denote the list of such car preferences by $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$, where $1 \leq \pi_i \leq n$ for every $1 \leq i \leq m$. We study the situation that $m \geq n$ and at least $k \leq n$ cars park successfully, i.e., there are at most $n-k$ unoccupied spots. In the special case $m = n = k$, this reduces to the classical situation where all cars get to park and all spots are occupied. Allowing $m > n \geq k$ introduces more complication to the parking scenario, as not all cars can park under any circumstances and there will always be cars that leave the street. These parking preferences are therefore referred to as "defective parking functions" (DPF) and are explored in depth by Cameron et al.[1]. Their results include counting defective parking functions DPF$(m,n,k)$, providing an equivalent expression given by Abel's binomial theorem, and studying the asymptotics. If $m > n = k$, then no spots are left unoccupied, so in some sense it still describes a "successful" parking situation. We thus denote DPF$(m,n,n)$ for brevity by SPF$(m,n)$. By taking the difference of having at least $k+1$ cars park successfully and at least $k$ cars park successfully, we can also identify the parking preferences that lead to exactly $k$ cars parking successfully.

The reader might be curious as to why we investigate the situation where at least $k$ cars park when there are $m$ cars and $n$ spots rather than directly investigate under what condition exactly $k$ cars park; the following proposition explains why by relating the parking preferences to $\mathbf{u}$-parking functions. Given a positive integer-valued vector $\mathbf{u} = (u_1, \ldots, u_m)$ with $u_1 \leq \cdots \leq u_m$, a $\mathbf{u}$-parking function of length $m$ is a sequence $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ of positive integers whose non-decreasing rearrangement $(\lambda_1, \ldots, \lambda_m)$ satisfies $\lambda_i \leq u_i$ for all $1 \leq i \leq m$. We denote the set of $\mathbf{u}$-parking functions by PF$(\mathbf{u})$.

**Proposition 1.1.** *Take a sequence of positive integers* $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ *with* $1 \leq \pi_i \leq n$ *for all* $1 \leq i \leq m$. *Then* $\boldsymbol{\pi} \in DPF(m,n,k)$ *if and only if* $\boldsymbol{\pi} \in PF(\mathbf{u})$, *where* $\mathbf{u} = (n-k+1, n-k+2, \ldots, n, \ldots, n)$ *is a vector of length* $m$. *Explicitly,* $\mathbf{u}$ *is a concatenation of an increasing arithmetic* *sequence of length k starting from $n-k+1$ and ending at n and a constant sequence of length $m-k$ where each term takes value n.*

*Remark* 1. Note that Proposition 1.1 implies that the set of defective parking functions $\mathrm{DPF}(m,n,k)$ is invariant under the action of $\mathfrak{S}_m$ by permuting cars. This proposition also gives a criterion for defective parking functions.

*Remark* 2. To show this equivalence, we will use the pigeonhole principle to show that for a given $\mathrm{DPF}(m,n,k)$ there is a $\mathbf{u}$-parking function that will produce the same parking result.

*Proof.* For cars that do not park, their preference can be any of the $n$ available spots on the street. But for cars that are able to park, their preference needs to satisfy the pigeonhole principle. Having at least $k$ spots taken is equivalent to ensuring that among those cars that park, we have at most one car prefer spot $n$, at most two cars prefer spot $\geq n-1$, and for each $i \leq k$ at most $i$ cars prefer spot $\geq n-i+1$. This combined condition for parked and unparked cars is further equivalent to having at least $k-1$ cars prefer spot $\leq n-1$, at least $k-2$ cars prefer spot $\leq n-2$, and at least one car prefer spot $\leq n-k+1$. Forming the non-decreasing rearrangement $(\lambda_1,\ldots,\lambda_m)$ of the parking preference $(\pi_1,\ldots,\pi_m)$ involves a switch of coordinates from $(i, \#\{j : \pi_j \leq i\})$ to $(\lambda_j, j)$ and gives the $\mathbf{u}$-parking function criterion. □

This paper is organized as follows: Section 2 illustrates the notion of parking function multi-shuffle that decomposes a defective parking function into smaller components (Definition 2.3 and Theorem 2.5); Section 3 uses the multi-shuffle construction introduced in Section 2 to investigate parking statistics of a defective parking function chosen uniformly at random. We give exact and asymptotic formulas for the distribution of the first parking coordinate in Proposition 3.1 and Theorem 3.2. Due to permutation symmetry, the result may be interpreted for any parking coordinate. Section 4 provides some further directions for research.

### 1.1 Notations

Let $\mathbb{N}$ be the set of non-negative integers. For $m,n \in \mathbb{N}$, we write $[m,n]$ for the set of integers $\{m,\ldots,n\}$ and $[n]=[1,n]$. For vectors $\mathbf{u},\mathbf{v} \in [n]^m$, denote by $\mathbf{u} \leq_C \mathbf{v}$ if $u_i \leq v_i$ for all $i \in [m]$; this is the component-wise partial order on $[n]^m$. In a similar fashion, denote by $\mathbf{u} <_C \mathbf{v}$ if $u_i \leq v_i$ for all $i \in [m]$ and there is at least one $j \in [m]$ such that $u_j < v_j$. For $\mathbf{v} \in [n]^m$, we write $[\mathbf{v}]$ for the set of $\mathbf{u} \in [n]^m$ with $\mathbf{u} \leq_C \mathbf{v}$.

*Remark* 3. With Proposition 1.1, we know that parking functions are shuffle invariant, so any ordering will produce the same result. The set $[n]^m$ refers to all reorderings of the arithmetic series $[1\ldots n]$, which we have shown to be equivalent to the non-decreasing ordering $\{1,2,\ldots,n\}$.

*Remark* 4. We will use $[v]$ to refer to the set of vectors that produce the same parking result as $v$, so any results that apply to $v$ will apply to those vectors which are component-wise less or equal.

**Example 1.2**

*Let* $\mathbf{v} \in$ *DPF(6, 6, 1),* $\mathbf{v}$ *= (1, 3, 3, 5, 6, 6). The fact that* $\mathbf{v}$ *is a parking function can be recovered by using Proposition 1.1. The set* $[\mathbf{v}]$ *will contain u = (1, 1, 1, 5, 5, 5) as well as u = (1, 3, 3, 5, 5, 6).*

## 2 PARKING FUNCTION MULTI-SHUFFLE

From Proposition 1.1, we see that in order to characterize the structure of parking preferences in $\mathrm{DPF}(m,n,k)$, we will be primarily concerned with examining $\mathbf{u}$-parking functions where $\mathbf{u}$ is a concatenation of an increasing arithmetic sequence and a constant sequence, and either of these sequences may be empty in the extreme case ($k=1$ and $m=k$ respectively). In this section, we explore the properties of such $\mathbf{u}$-parking functions through a parking function multi-shuffle construction. The construction is an extension to the framework studied in Yin[8] where generic $\mathbf{u}$-parking functions $\mathbf{u}=(u_1,\ldots,u_m)$ with $u_i < u_{i+1}$ for every $i$ was considered.

*Remark* 5. Multi-shuffles are a construction in which some of the elements of the parking vector are given, and the rest of the vector can be filled in with some possibilities based on the number of successes $k$ given. The multi-shuffle is the set of possible values that would satisfy $k$. This construction allows us to study more general Defective Parking Functions by working with a family of Parking Functions instead of one explicit vector.

We will write our results in terms of parking coordinates $\pi_1,\ldots,\pi_l$ for explicitness, where $1 \leq l \leq m$ is any integer. But due to permutation symmetry established in Proposition 1.1, they may be interpreted for any coordinates. Temporarily fix $\pi_{l+1},\ldots,\pi_m$. Let

$$A_{\pi_{l+1},\ldots,\pi_m} = \{\mathbf{v}=(v_1,\ldots,v_l) : (v_1,\ldots,v_l,\pi_{l+1},\ldots,\pi_m) \in \mathrm{DPF}(m,n,k)\}, \tag{2.1}$$

where $\mathbf{v}$ is in non-decreasing order.

**Proposition 2.1.** *Fix* $\pi_{l+1},\ldots,\pi_m$*. Then* $A_{\pi_{l+1},\ldots,\pi_m} = [\mathbf{v}]$ *for a unique* $\mathbf{v}$*.*

*Remark* 6. Here we are fixing the last $m-l$ elements of the vector. Then we show that $[\mathbf{v}]$ is exactly the set of vectors which when combined with $(\pi_{l+1},\dots,\pi_m)$ will be in the set of $\mathrm{DPF}(m,n,k)$.

*Proof.* From the equivalence shown in Proposition 1.1, it is sufficient to display the unique maximal $\mathbf{v}$ so that $(v_1,\dots,v_l,\pi_{l+1},\dots,\pi_m) \in \mathrm{PF}(\mathbf{u})$, where $\mathbf{u} = (n-k+1,n-k+2,\dots,n,\dots,n)$ is a vector of length $m$. We arrange $\pi_i$ for $l+1 \le i \le m$ in non-decreasing order, denoted by $\pi_{(l+1)} \le \cdots \le \pi_{(m)}$. Set $n_l = 0$. We find the minimum index $n_i$ in order, starting with $n_{l+1}$, such that $n_i > n_{i-1}$ and $u_{n_i} \ge \pi_{(i)}$ for each $l+1 \le i \le m$. If such $u_{n_i}$'s cannot be located, then $A_{\pi_{l+1},\dots,\pi_m}$ is empty. Otherwise excluding these $u_{n_i}$'s from $\mathbf{u}$ gives the optimal $\mathbf{v}$. From the parking scheme, if $\mathbf{v} \in A_{\pi_{l+1},\dots,\pi_m}$, then $\mathbf{w} \in A_{\pi_{l+1},\dots,\pi_m}$ for all $\mathbf{w} \le_C \mathbf{v}$, where $\le_C$ is the component-wise partial order. This implies that if $A_{\pi_{l+1},\dots,\pi_m}$ is non-empty, then there is a unique maximal element in component-wise partial order $\mathbf{v}$ in $A_{\pi_{l+1},\dots,\pi_m}$, when we require that $\mathbf{v}$ is arranged in non-decreasing order. □

**Example 2.2.** *Take $m=8$, $n=6$, and $k=4$. Consider $DPF(8,6,4)$ with associated $\mathbf{u} = (u_1,\cdots,u_8) = (3,4,5,6,6,6,6,6)$. Take $l=5$ and set $\pi_6 = 6$, $\pi_7 = 4$ and $\pi_8 = 6$. Then $A_{\pi_6,\pi_7,\pi_8} = [\mathbf{v}] = [(u_1,u_3,u_6,u_7,u_8)] = [(3,5,6,6,6)]$. See illustration below.*

| | | | | |
|---|---|---|---|---|
| $v_1$ | | | 3 | $u_1$ |
| $\pi_{(6)}$ | 4 | $\le$ | 4 | $u_2$ |
| $v_2$ | | | 5 | $u_3$ |
| $\pi_{(7)}$ | 6 | $\le$ | 6 | $u_4$ |
| $\pi_{(8)}$ | 6 | $\le$ | 6 | $u_5$ |
| $v_3$ | | | 6 | $u_6$ |
| $v_4$ | | | 6 | $u_7$ |
| $v_5$ | | | 6 | $u_8$ |

We conclude from Proposition 2.1 and Example 2.2 that when the last $m-l$ parking preferences of an element of $\mathrm{DPF}(m,n,k)$ are given, we only need to find the largest feasible first $l$ parking preferences. Correspondingly, we introduce a combinatorial construction which we term a parking function multi-shuffle to $\mathbf{u}$-parking functions of the form $\mathbf{u} = (n-k+1,n-k+2,\dots,n,\dots,n)$. This construction will connect the identification of the maximal element in $A_{\pi_{l+1},\dots,\pi_m}$ to the decomposition of $\pi_{l+1},\dots,\pi_m$ into a multi-shuffle.

**Definition 2.3** (parking function multi-shuffle)**.** *Take $1 \le k \le n \le m$ and $1 \le l \le m$.*

- *(Generic mixed case) Let $\mathbf{v} = (v_1,\dots,v_l) \in [n]^l$ be such that $n-k+1 \le v_1 < \cdots < v_r < v_{r+1} = \cdots = v_l = n$ for some $r$ with $r \le k-1$ and $l-r \le m-k+1$. Say that $\pi_{l+1},\dots,\pi_m$ is a* parking function multi-shuffle *of $r+1$ $\mathbf{u}$-parking functions $\boldsymbol{\alpha}_1 \in PF(n-k+1,n-k+2,\dots,v_1-1)$, $\boldsymbol{\alpha}_2 \in PF(1,2,\dots,v_2-v_1-1),\dots,\boldsymbol{\alpha}_r \in PF(1,2,\dots,v_r-v_{r-1}-1)$, and $\boldsymbol{\alpha}_{r+1} \in PF(1,2,\dots,n-v_r,\dots,n-v_r)$ if $\pi_{l+1},\dots,\pi_m$ is any permutation of the union of the $r+1$ words $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 + (v_1,\dots,v_1),\dots,\boldsymbol{\alpha}_{r+1} + (v_r,\dots,v_r)$. (Some $\boldsymbol{\alpha}_j$ might be empty.) We will denote this by $(\pi_{l+1},\dots,\pi_m) \in MS(\mathbf{v})$.*
- *(Special case: increasing arithmetic sequence) Let $\mathbf{v} = (v_1,\dots,v_l) \in [n]^l$ be such that $n-k+1 \le v_1 < \cdots < v_l < n$ with $l \le k-1$. Say that $\pi_{l+1},\dots,\pi_m$ is a* parking function multi-shuffle *of $l+1$ $\mathbf{u}$-parking functions $\boldsymbol{\alpha}_1 \in PF(n-k+1,n-k+2,\dots,v_1-1)$, $\boldsymbol{\alpha}_2 \in PF(1,2,\dots,v_2-v_1-1),\dots,\boldsymbol{\alpha}_l \in PF(1,2,\dots,v_l-v_{l-1}-1)$, and $\boldsymbol{\alpha}_{l+1} \in PF(1,2,\dots,n-v_l,\dots,n-v_l)$ if $\pi_{l+1},\dots,\pi_m$ is any permutation of the union of the $l+1$ words $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 + (v_1,\dots,v_1),\dots,\boldsymbol{\alpha}_{l+1} + (v_l,\dots,v_l)$. (Some $\boldsymbol{\alpha}_j$ might be empty.) We will denote this by $(\pi_{l+1},\dots,\pi_m) \in MS(\mathbf{v})$.*

**Example 2.4.** *Take $m=10$, $n=8$, $k=6$, and $l=3$.*

- *(Generic mixed case) Set $v_1 = 4$, $v_2 = 6$, and $v_3 = 8$. Take $\boldsymbol{\alpha}_1 = (3) \in PF(3)$, $\boldsymbol{\alpha}_2 = (1) \in PF(1)$, and $\boldsymbol{\alpha}_3 = (2,1,1,1,2) \in PF(1,2,2,2,2)$. Then $(7,5,3,8,7,8,7) \in MS(4,6,8)$ is a multi-shuffle of the three words $(3)$, $(5)$, and $(8,7,7,7,8)$.*
- *(Special case: increasing arithmetic sequence) Set $v_1 = 3$, $v_2 = 5$, and $v_3 = 7$. Take $\boldsymbol{\alpha}_1 = \emptyset$, $\boldsymbol{\alpha}_2 = (1) \in PF(1)$, $\boldsymbol{\alpha}_3 = (1) \in PF(1)$, and $\boldsymbol{\alpha}_4 = (1,1,1,1,1) \in PF(1,1,1,1,1)$. Then $(8,8,6,4,8,8,8) \in MS(3,5,7)$ is a multi-shuffle of the four words $\emptyset$, $(4)$, $(6)$, and $(8,8,8,8,8)$.*

**Theorem 2.5.** *Take $1 \le k \le n \le m$ and $1 \le l \le m$. Let $\mathbf{v} = (v_1,\dots,v_l) \in [n]^l$ be in non-decreasing order as in Definition 2.3. Then $A_{\pi_{l+1},\dots,\pi_m} = [\mathbf{v}]$ if and only if $(\pi_{l+1},\dots,\pi_m) \in MS(\mathbf{v})$.*

*Remark* 7. Here we are expanding on the result from Proposition 2.1 by allowing $(\pi_{l+1},\dots,\pi_m)$ to be non-consecutive, we need the extra requirement that $(\pi_{l+1},\dots,\pi_m)$ is in the set of multi-shuffles for $\mathbf{v}$.

*Remark* 8. For the special case where $\mathbf{v}$ is a constant sequence with $v_1 = \cdots = v_l = n$ and $l \le m-k+1$, $A_{\pi_{l+1},\dots,\pi_m} = [\mathbf{v}]$ if and only if $(\pi_{l+1},\dots,\pi_m) \in \mathrm{PF}(\mathbf{u})$, where $\mathbf{u} = (n-k+1,n-k+2,\dots,n,\dots,n)$ is a vector of length $m-l$. Trivially, it is a multi-shuffle of only one word.

*Proof.* Following the equivalence established in Proposition 1.1, set $\mathbf{u} = (n-k+1, n-k+2, \ldots, n, \ldots, n)$ to be a vector of length $m$.

"$\Longrightarrow$" Take $\boldsymbol{\pi} = (v_1, \ldots, v_l, \pi_{l+1}, \ldots, \pi_m)$ a $\mathbf{u}$-parking function, where $\mathbf{v}$ is maximally compatible with the fixed $\pi_{l+1}, \ldots, \pi_m$. Therefore we must have $v_i \geq n-k+1$ and $v_i = u_{v_i-n+k}$ for every $1 \leq i \leq l$ since otherwise the value of $v_i$ may be increased, contradicting the maximality assumption.

Hence excluding the first $l$ cars, $\boldsymbol{\pi}$ has exactly $v_1 - n + k - 1$ cars with preference $\leq v_1 - 1$ (name the subsequence $\boldsymbol{\alpha}_1$), exactly $v_2 - v_1 - 1$ cars with preference $\geq v_1 + 1$ and $\leq v_2 - 1$ (name the subsequence $\boldsymbol{\alpha}_2'$), .... Construct $\boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_2' - (v_1, \ldots, v_1), \boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_3' - (v_2, \ldots, v_2), \ldots$. It is clear from the above reasoning that $\boldsymbol{\alpha}_1 \in \mathrm{PF}(n-k+1, n-k+2, \ldots, v_1 - 1)$, $\boldsymbol{\alpha}_2 \in \mathrm{PF}(1, 2, \ldots, v_2 - v_1 - 1)$, .... By Definition 2.3, $(\pi_{l+1}, \ldots, \pi_m) \in \mathrm{MS}(\mathbf{v})$.

"$\Longleftarrow$" We first show that $\boldsymbol{\pi} = (v_1, \ldots, v_l, \pi_{l+1}, \ldots, \pi_m)$ is a $\mathbf{u}$-parking function. This is immediate, since from Definition 2.3, the non-decreasing rearrangement of $\boldsymbol{\pi}$ is a concatenation of $\boldsymbol{\alpha}_1, v_1, \boldsymbol{\alpha}_2 + (v_1, \ldots, v_1), v_2, \ldots$.

Next we show that $\boldsymbol{\pi}^i = (v_1, \ldots, v_{i-1}, v_i + 1, v_{i+1}, \ldots, v_l, \pi_{l+1}, \ldots, \pi_m)$ is not a $\mathbf{u}$-parking function for any $1 \leq i \leq l$. This is clear when $v_i = n$. Suppose $v_i < n$, then the non-decreasing rearrangement of $\boldsymbol{\pi}^i$ only differs from the non-decreasing rearrangement of $\boldsymbol{\pi}$ in the $(v_i - n + k)$-th position with value $v_i + 1 > v_i = u_{v_i-n+k}$.

Combining, we have $A_{\pi_{l+1}, \ldots, \pi_m} = [\mathbf{v}]$. □

*Remark* 9. Even though multi-shuffle is not explicitly mentioned in the main proof of Theorem 3.2, being able to construct multi-shuffles on parking functions is a very important technique that allows for the counting of parking functions.

**Example 2.6** (Continued from Example 2.4)**.** *Take $m = 10$, $n = 8$, and $k = 6$. Consider $DPF(10, 8, 6)$ with associated $\mathbf{u} = (3, 4, 5, 6, 7, 8, 8, 8, 8, 8)$. Then $A_{7,5,3,8,7,8,7} = [(4, 6, 8)]$ is equivalent to $(7, 5, 3, 8, 7, 8, 7) \in MS(4, 6, 8)$ and $A_{8,8,6,4,8,8,8} = [(3, 5, 7)]$ is equivalent to $(8, 8, 6, 4, 8, 8, 8) \in MS(3, 5, 7)$. See illustration below.*

$$
\begin{array}{ccccc}
\pi_{(4)} & 3 & \leq & 3 & u_1 \\
v_1 & & & 4 & u_2 \\
\pi_{(5)} & 5 & \leq & 5 & u_3 \\
v_2 & & & 6 & u_4 \\
\pi_{(6)} & 7 & \leq & 7 & u_5 \\
\pi_{(7)} & 7 & \leq & 8 & u_6 \\
\pi_{(8)} & 7 & \leq & 8 & u_7 \\
\pi_{(9)} & 8 & \leq & 8 & u_8 \\
\pi_{(10)} & 8 & \leq & 8 & u_9 \\
v_3 & & & 8 & u_{10}
\end{array}
\qquad\qquad
\begin{array}{ccccc}
v_1 & & & 3 & u_1 \\
\pi_{(4)} & 4 & \leq & 4 & u_2 \\
v_2 & & & 5 & u_3 \\
\pi_{(5)} & 6 & \leq & 6 & u_4 \\
v_3 & & & 7 & u_5 \\
\pi_{(6)} & 8 & \leq & 8 & u_6 \\
\pi_{(7)} & 8 & \leq & 8 & u_7 \\
\pi_{(8)} & 8 & \leq & 8 & u_8 \\
\pi_{(9)} & 8 & \leq & 8 & u_9 \\
\pi_{(10)} & 8 & \leq & 8 & u_{10}
\end{array}
$$

## 3 PARKING STATISTICS

In this section, we use the multi-shuffle construction introduced in Section 2 to investigate parking statistics of random defective parking functions and identify a sharp transition. We will utilize some counting formulas from Cameron et al[1] where it was shown that

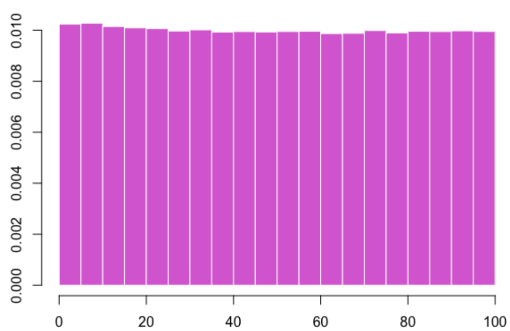
$$
\begin{aligned}
|\,\mathrm{DPF}(m, n, k)| &= (n-k+1) \sum_{j=k}^{m} \binom{m}{j} (n-k+j+1)^{j-1} (k-j-1)^{m-j} \\
&= n^m - (n-k+1) \sum_{j=0}^{k-1} \binom{m}{j} (n-k+j+1)^{j-1} (k-j-1)^{m-j},
\end{aligned}
\tag{3.1}
$$

and in particular,

$$
|\,\mathrm{SPF}(m, n)| = \sum_{j=n}^{m} \binom{m}{j} (j+1)^{j-1} (n-j-1)^{m-j} = n^m - \sum_{j=0}^{n-1} \binom{m}{j} (j+1)^{j-1} (n-j-1)^{m-j}. \tag{3.2}
$$

Here the equivalence of the expressions is due to Abel's binomial theorem. Recall that parking coordinates satisfy permutation symmetry, so the statements in this section may be interpreted for any parking coordinates and not just the first parking coordinate.

**Proposition 3.1.** *Take $m > n \geq k$. The number of defective parking functions $\boldsymbol{\pi} \in DPF(m, n, k)$ with $\pi_1 = j$ is*



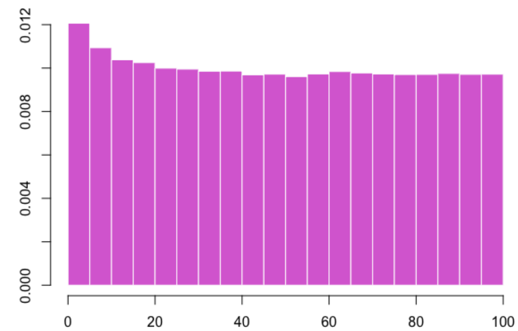




**Figure 1.** The distribution of $\pi_1$ (the first parking coordinate) in 100,000 samples of defective parking functions chosen uniformly at random from DPF(120, 100, 95) (left plot) and DPF(120, 100, 99) (right plot).

*Remark* 10. We were most interested in the behavior for $\pi_1 = 1$ because, as the distributions show, the asymptotic behavior is more present at smaller values

$$\sum_{v=\max(j,n-k+1)}^{n-1} \binom{m-1}{n-v+m-k}(n-k+1)v^{k-n+v-2}\cdot$$
$$\cdot\left[(n-v)^{n-v+m-k} - \sum_{s=0}^{n-v-1}\binom{n-v+m-k}{s}(s+1)^{s-1}(n-v-s-1)^{n-v+m-k-s}\right]$$
$$+\left[n^{m-1} - (n-k+1)\sum_{s=0}^{k-1}\binom{m-1}{s}(n-k+s+1)^{s-1}(k-s-1)^{m-s-1}\right]. \tag{3.3}$$

*Proof.* From the parking scheme, if $\pi_1 = j$, then $A_{\pi_2,\ldots,\pi_m} = [v]$ for some $\max(j, n-k+1) \le v \le n$. By Theorem 2.5 and utilizing the equivalence between **u**-parking functions and defective parking functions derived in Proposition 1.1, we have the number of defective parking functions with $\pi_1 = j$ is

$$\sum_{v=\max(j,n-k+1)}^{n-1} \binom{m-1}{n-v+m-k} |\,\mathrm{DPF}(k-n+v-1, v-1, k-n+v-1)|\cdot$$
$$\cdot\,|\,\mathrm{SPF}(n-v+m-k, n-v)|$$
$$+\,|\,\mathrm{DPF}(m-1,n,k)|. \tag{3.4}$$

The conclusion readily follows. See Figure 1. □

**Theorem 3.2.** *Take $m$ and $n$ large with $m = cn$ for some $c > 1$. For a defective parking function $\boldsymbol{\pi}$ chosen uniformly at random from $SPF(m,n)$, we have*

$$\mathbb{E}(\pi_1) = \frac{\sum_{j=1}^{n} j\#\{\boldsymbol{\pi} \in SPF(m,n) : \pi_1 = j\}}{|\,SPF(m,n)|} = \frac{n}{2}\left(1 + \frac{1}{n}\left(1 - \frac{d}{c(1-d)}\right) + O\left(\frac{1}{n^2}\right)\right), \tag{3.5}$$

*where $d$ is the unique solution in $(0,1)$ satisfying $ce^{-c} = de^{-d}$. In particular, $d = 1$ if $c = 1$, and $d \to 0$ if $c \to \infty$.*

*Remark* 11. $ce^{-c} = de^{-d}$ is a Lambert W Function. This function has two roots, one greater than 1 and one between 0 and 1. We know $c$ from $m = cn$, so the other root d must be less than 1. When we have found these two roots, the formula is quite good for finding the expected distribution.

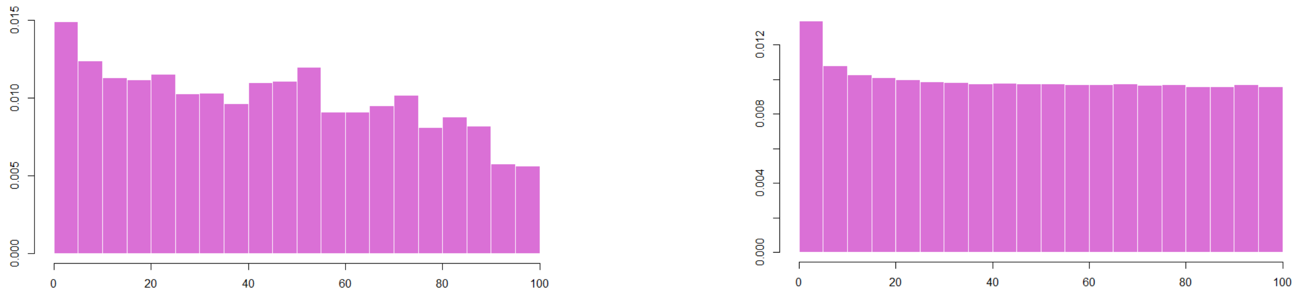



**Figure 2.** The distribution of $\pi_1$ (the first parking coordinate) in 100,000 samples of defective parking functions chosen uniformly at random from SPF(100, 100) (left plot) and SPF(120, 100) (right plot).

*Remark* 12. Take $m = n$ large, it was derived in Kenyon & Yin[9] that

$$\mathbb{E}(\pi_1) = \frac{n}{2}\left(1 - \sqrt{\frac{\pi}{2n}} + \frac{10}{3n} + O(n^{-3/2})\right). \tag{3.6}$$

We observe a sharp change as $c \to 1$ in the next leading order term of the moment asymptotics. See Figure 2. In this notation, $O(n^{-3/2})$ means all the preceding terms are omitted as their contribution is relatively small.

*Remark* 13. The asymptotics in Equation (3.5) is quite accurate. Take $m = 240$ and $n = 200$ with $c = 1.2$. The exact value of $\mathbb{E}(\pi_1) = 98.8379$, while the asymptotic approximation gives 98.5551.

*Proof.* By (3.2), the denominator of (3.5) is

$$|\operatorname{SPF}(m,n)| = n^m - \sum_{j=0}^{n-1} \frac{m^j}{j!} n^{m-j} e^{-c(j+1)} (j+1)^{j-1} \cdot \left(1 - \frac{j(j-1)}{2cn} - \frac{c(j+1)^2}{2n} + \frac{j(j+1)}{n} + O\left(n^{-2}\right)\right). \tag{3.7}$$

The tree function $F(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}(j+1)^{j-1}$ is related to the Lambert $W$ function via $F(z) = -W(-z)/z$, and satisfies $F(ce^{-c}) = \frac{d}{c}e^c$, where $d$ is the unique solution in $(0,1)$ such that $ce^{-c} = de^{-d}$. By the chain rule,

$$F'(z) = \frac{F^2(z)}{(1 - zF(z))},$$

$$F'(ce^{-c}) = \frac{d^2}{c^2(1-d)} e^{2c}, \qquad F''(ce^{-c}) = \frac{d^3(3-2d)}{c^3(1-d)^3} e^{3c}. \tag{3.8}$$

We recognize that (3.7) is of the form

$$n^m \left[1 - e^{-c} \sum_{j=0}^{\infty} \frac{(ce^{-c})^j}{j!} (j+1)^{j-1} \left(1 + \frac{1}{n}(A_1 + A_2 j + A_3 j^2) + O\left(n^{-2}\right)\right)\right], \tag{3.9}$$

where

$$A_1 = -\frac{c}{2}, \quad A_2 = -c + \frac{1}{2c} + 1, \quad A_3 = -\frac{c}{2} - \frac{1}{2c} + 1. \tag{3.10}$$

Using $F(z)$ this can be written as (with $z = ce^{-c}$):

$$n^m \left[1 - e^{-c}\left(F(z) + \frac{1}{n}\left(A_1 F(z) + A_2 z F'(z) + A_3 (z^2 F''(z) + zF'(z))\right) + O\left(\frac{1}{n^2}\right)\right)\right]. \tag{3.11}$$

From Proposition 3.1, the numerator of (3.5) is the sum of $I_1$ and $I_2$, with

$$I_1 = \left[\sum_{j=1}^{n} j\right]\left[n^{m-1} - \sum_{s=0}^{n-1}\binom{m-1}{s}(s+1)^{s-1}(n-s-1)^{m-s-1}\right], \tag{3.12}$$

$$I_2 = \sum_{j=1}^{n-1} j \sum_{v=j}^{n-1}\binom{m-1}{m-v} v^{v-2}\left[(n-v)^{m-v} - \sum_{s=0}^{n-v-1}\binom{m-v}{s}(s+1)^{s-1}(n-v-s-1)^{m-v-s}\right]. \tag{3.13}$$

For $I_1$, we have

$$I_1 = \frac{n^2}{2}\left(1+\frac{1}{n}\right)\left[n^{m-1} - \sum_{s=0}^{n-1}\frac{m^s}{s!}n^{m-s-1}e^{-c(s+1)}(s+1)^{s-1}\cdot \right.$$
$$\left. \cdot\left(1 - \frac{s(s+1)}{2cn} - \frac{c(s+1)^2}{2n} + \frac{(s+1)^2}{n} + O\left(n^{-2}\right)\right)\right]. \tag{3.14}$$

We recognize that (3.14) is of the form

$$\frac{n^{m+1}}{2}\left(1+\frac{1}{n}\right)\left[1 - e^{-c}\sum_{s=0}^{\infty}\frac{(ce^{-c})^s}{s!}(s+1)^{s-1}\left(1+\frac{1}{n}(B_1+B_2 s+B_3 s^2) + O\left(n^{-2}\right)\right)\right], \tag{3.15}$$

where

$$B_1 = -\frac{c}{2}+1, \qquad B_2 = -c-\frac{1}{2c}+2, \qquad B_3 = -\frac{c}{2}-\frac{1}{2c}+1. \tag{3.16}$$

Using $F(z)$ this can be written as (with $z = ce^{-c}$):

$$\frac{n^{m+1}}{2}\left(1+\frac{1}{n}\right)\left[1 - e^{-c}\left(F(z) + \frac{1}{n}\left(B_1 F(z) + B_2 zF'(z) + B_3(z^2F''(z)+zF'(z))\right) + O\left(\frac{1}{n^2}\right)\right)\right]. \tag{3.17}$$

For $I_2$, we have

$$\begin{aligned}
I_2 &= \sum_{v=0}^{n-2}\binom{m-1}{v}(v+1)^{v-1}(n-v-1)^{m-v-1}\frac{(v+1)^2}{2}\left(1+\frac{1}{v+1}\right) \\
&- \sum_{v=0}^{n-2}\sum_{s=0}^{n-v-2}\binom{m-1}{v, s, m-1-v-s}(v+1)^{v-1}(s+1)^{s-1}(n-v-s-2)^{m-v-s-1}\frac{(v+1)^2}{2}\left(1+\frac{1}{v+1}\right) \\
&= \frac{1}{2}\sum_{v=0}^{n-2}\frac{m^v}{v!}n^{m-v-1}e^{-c(v+1)}\left((v+1)^{v+1}+(v+1)^v\right)\left(1+O\left(n^{-1}\right)\right) \\
&\quad - \frac{1}{2}\sum_{v=0}^{n-2}\sum_{s=0}^{n-v-2}\frac{m^{v+s}}{v!s!}n^{m-v-s-1}e^{-c(v+s+2)}\left((v+1)^{v+1}+(v+1)^v\right)(s+1)^{s-1}\left(1+O\left(n^{-1}\right)\right).
\end{aligned} \tag{3.18}$$

The generalized tree functions $G(z) = \sum_{s=0}^{\infty}\frac{z^s}{s!}(s+1)^s = \left(\sum_{s=0}^{\infty}\frac{z^s}{s!}s^{s-1}\right)'$ and $H(z) = \sum_{s=0}^{\infty}\frac{z^s}{s!}(s+1)^{s+1} = \left(\sum_{s=0}^{\infty}\frac{z^s}{s!}(s-1)^{s-1}\right)''$ are related to the tree function $F(z)$, and respectively satisfy

$$G(ce^{-c}) = \frac{d}{c(1-d)}e^c, \qquad H(ce^{-c}) = \frac{d}{c(1-d)^3}e^c. \tag{3.19}$$

We recognize that (3.18) is of the form

$$\begin{aligned}
&\frac{1}{2}n^{m-1}e^{-c}\sum_{v=0}^{\infty}\frac{(ce^{-c})^v}{v!}\left((v+1)^{v+1}+(v+1)^v\right)\left(1+O\left(n^{-1}\right)\right) \\
&\quad - \frac{1}{2}n^{m-1}e^{-2c}\sum_{v=0}^{\infty}\sum_{s=0}^{\infty}\frac{(ce^{-c})^{v+s}}{v!s!}\left((v+1)^{v+1}+(v+1)^v\right)(s+1)^{s-1}\left(1+O\left(n^{-1}\right)\right).
\end{aligned} \tag{3.20}$$

Using $F(z)$, $G(z)$, and $H(z)$ this can be written as (with $z = ce^{-c}$ and $I(z) = G(z) + H(z)$):

$$\frac{1}{2}n^{m-1}e^{-c}\left[I(z) + O\left(\frac{1}{n}\right)\right] - \frac{1}{2}n^{m-1}e^{-2c}\left[I(z)F(z) + O\left(\frac{1}{n}\right)\right]. \tag{3.21}$$

Dividing (3.11) into (3.17)+(3.21) and simplifying we get our desired result. □

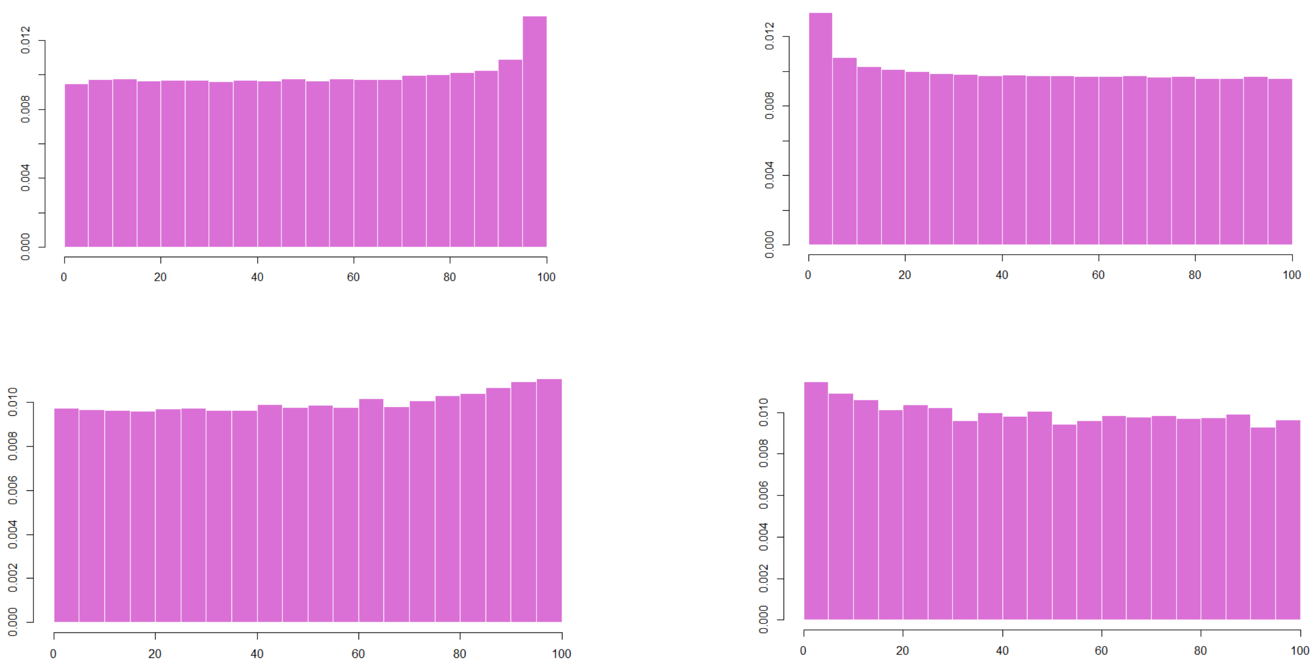



**Figure 3.** The distribution of $\pi_1$ (the first parking coordinate) in 100,000 samples of defective parking functions chosen uniformly at random from SPF$(120, 100)$. The upper left plot is for $p = 0$ and the upper right plot is for $p = 1$. The lower left plot is for $p = 0.1$ and the lower right plot is for $p = 0.9$. Note the preference symmetry between $p$ and $1 - p$. $p = 1$ corresponds to the deterministic parking protocol studied in this paper.

## 4 FURTHER RESEARCH

In Durmic et al[10], a probabilistic parking protocol was considered, which added one more layer of complexity to the parking scenario. Fix $p \in [0,1]$ and consider a coin that flips to heads with probability $p$ and tails with probability $1-p$. If a car arrives at its preferred spot and finds it unoccupied it parks there. If instead the spot is occupied, then the driver tosses the biased coin. If the coin lands on heads, with probability $p$, the driver continues moving forward in the street. However, if the coin lands on tails, with probability $1-p$, the car moves backward and tries to find an unoccupied parking spot. We see that the deterministic parking protocol where the car always moves forward if its desired spot is taken corresponds to $p = 1$.

Only the effect of the probabilistic protocol on the classical parking situation ($m = n = k$) was investigated in Durmic et al[10]. The authors are further interested in similarly researching the probabilistic effect on defective parking functions. See Figure 3 for some initial simulations.

## 5 CONCLUSION

The focus of our paper was to analyze the asymptotic behavior of Defective Parking Function distributions. First, we looked at the equivalence of DPF$(m,n,k)$ to some $u$-parking functions, which allowed us to more easily analyze DPFs. We also analyzed the multi-shuffles of DPFs and exhibited an algorithm for building multi-shuffles. The main results were in counting the number of Defective Parking Functions as well as the distribution of certain preferences which exhibited the asymptotic behavior that we see in the graphs.

## 6 AUTHOR'S NOTE

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## 7 EDITOR'S NOTES

This article was peer-reviewed.

## REFERENCES

[1] Cameron, P. J., Johannsen, D., Prellberg, T. & Schweitzer, P. Counting defective parking functions. *Electronic Journal of Combinatorics* **15** (2008).

[2] Konheim, A. G. & Weiss, B. An Occupancy Discipline and Applications. *SIAM Journal on Applied Mathematics* **14** (1966).

[3] Chassaing, P. & Marckert, J. F. Parking functions, empirical processes, and the width of rooted labeled trees. *Electronic Journal of Combinatorics* **8** (2001).

[4] Stanley, R. P. Parking functions and noncrossing partitions. *Electronic Journal of Combinatorics* **4** (1997).

[5] Stanley, R. P. Hyperplane Arrangements, Parking Functions and Tree Inversions. In *Mathematical Essays in honor of Gian-Carlo Rota*, vol. 161, 359–375 (Birkhauser, Boston, 1998).

[6] Cori, R. & Rossin, D. On the Sandpile Group of Dual Graphs. *European Journal of Combinatorics* **21** (2000).

[7] Yan, C. Parking functions. *Bona, M. (ed.) Handbook of Enumerative Combinatorics. Discrete Math. Appl.* 835–893 (2015).

[8] Yin, M. Parking Functions, Multi-Shuffle, and Asymptotic Phenomena. In Ward, M. D. (ed.) *33rd International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2022)*, vol. 225 of *Leibniz International Proceedings in Informatics (LIPIcs)*, 18:1–18:12 (Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 2022). URL https://drops.dagstuhl.de/opus/volltexte/2022/16104.

[9] Kenyon, R. & Yin, M. Parking functions: From combinatorics to probability (2021). URL https://arxiv.org/abs/2103.17180.

[10] Durmić, I., Han, A., Harris, P. E., Ribeiro, R. & Yin, M. Probabilistic parking functions (2022). URL https://arxiv.org/abs/2211.00536.